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Excess angular momentum in n dimensions

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Abstract. A modified form of excess angular momentum is introduced into both the n dimensional hydrogen atom and the harmonic oscillator. The degeneracies of the new energy levels are shown to be simply related to the representations of $O(n+1)$ and $SU(n)$, and in one case this is explained by decomposing the excess system into the direct sum of similar normal systems.

1. Introduction

Excess angular momentum was first introduced by Cisneros and McIntosh (1970) when they were considering the problem of relating a universal symmetry group to the degeneracies of two-dimensional quantum-mechanical systems. The total angular momentum was increased by multiplying the angular momentum operator with a constant factor, and they examined the effects of this on the degeneracy.

Such systems are somewhat artificial although when the multiplying factor is an integer the modification is equivalent to restricting the values of the angular momentum to multiples of an integer. Even when the multiplying factor is not an integer there are several reasons for considering what happens when excess angular momentum is introduced into n dimensional systems.

Firstly, the presence of extra (or 'accidental') degeneracy in the normal cases is not clearly understood and universal symmetry groups have been proposed for such n dimensional problems, so that, like Cisneros and McIntosh, we wish to discover the effect of making minor alterations to these problems. Although Cisneros and McIntosh (1970) mention the n dimensional cases, their results are not consistent and so it is important to correct and complete their work. In fact, in some respects, it is easier to consider adding excess angular momentum, in a modified form, to the n dimensional systems than it was to the two-dimensional ones and it is interesting to discover that there are precise formulae relating the degeneracies in several different cases.

However, the main motivation for this work arises from the case when the multiplying factor is an integer. This is a more realistic problem and possesses some special properties similar to those of the anisotropic harmonic oscillator. As explained in King (1973) several authors have studied the degeneracies of the anisotropic oscillator and so it is very interesting to have another example which leads to an almost identical situation. The importance of both these examples is that they challenge the usual explanations given for the 'accidental' degeneracy of the oscillator and hydrogen atom. Although we have produced decompositions which account for the degeneracies there is still no

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reason for expecting these modified systems to possess accidental degeneracy and good theories ought to be able to predict the degeneracies for these cases as well as the normal ones.

In § 2 a modified form of excess angular momentum is introduced and the eigenvalues noted. The degeneracies are given in § 3, and in § 4 these are related to the patterns occurring in the normal cases. Finally, it is shown how an excess system with an integer multiplying factor can be decomposed in a way which accounts for the degeneracies.

2. Excess systems

The two systems to be considered are the n dimensional, quantum-mechanical, hydrogen atom and isotropic oscillator, as in both cases the Schrödinger equation of the problem can be written,

$$\left[r^{1-n}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\right)-r^{-2}L_n^2-V(r)\right]\psi = E\psi, \tag{1}$$

where L_n^2 is the total orbital momentum operator in n dimensions and $V(r)$ takes the forms $-r^{-1}$ and r^2 respectively.

As L_n^2 is independent of r, the usual method of solving (1) is 'separation of the variables' and so ψ is written $\psi = f_m(r)Y_m$, where Y_m is an eigenfunction of L_n^2. The function $f_m(r)$ is any solution of (1) with the appropriate eigenvalues of L_n^2 substituted in place of L_n^2.

It has been shown by Joseph (1967) that L_n^2 takes the values $m(m+n-2)$ for all non-negative integers m, and that the eigenfunctions have O(n) symmetry. In fact, for $n \geqslant 2$, Joseph shows that the degeneracy, $D_n(m)$, of the eigenvalue $m(m+n-2)$ is given by

$$\begin{aligned} D_n(m) &= (m+n-3)!(2m+n-2)[(n-2)!m!]^{-1} \\ &= C_m^{m+n-1}-C_{m-3}^{m+n-3} \qquad (m \geqslant 2). \end{aligned} \tag{2}$$

Now it can be seen that $f_m(r)$ is a solution of

$$\left[r^{1-n}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{n-1}\frac{\mathrm{d}}{\mathrm{d}r}\right)-m(m+n-2)r^{-2}-V(r)\right]f_m = Ef_m \tag{1a}$$

and any solution of (1) has at least O(n) degeneracy due to the degeneracy of the angular momentum. However, it turns out that there is extra, 'accidental' degeneracy due to the nature of the solutions of (1a). We are interested in examining what happens to this extra degeneracy when the equation (1a) is slightly modified.

Cisneros and McIntosh (1970) modify this equation by introducing the positive scalar factor c^2, which multiplies each eigenvalue of L_n^2. However, in some ways it is more natural to consider what happens when each integer m is multiplied by a positive constant. If this constant is an integer the modification is equivalent to restricting m to all the multiples of that integer.

Thus, in this paper, the hydrogen atom and isotropic oscillator with excess angular momentum will be described by the functions $\psi = f_m(r)Y_m$, such that Y_m is an eigenfunction of L_n^2 and $f_m(r)$ is a solution of

$$\left[r^{1-n}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{n-1}\frac{\mathrm{d}}{\mathrm{d}r}\right)-cm(cm+n-2)r^{-2}-V(r)\right]f_m = Ef_m \tag{3}$$

where c is a positive constant and E is the energy of the system. As we are only modifying

the equation determining f_m these systems still possess O(n) symmetry due to the eigenfunctions Y_m and so for each non-negative integer m the degeneracy is still given by (2).

Equation (3) can be solved using any of the methods employed when $c = 1$ (cf Bander and Itzykson 1966 or Schrödinger 1940), and if E_1 and E_2 denote the energies of the hydrogen atom and oscillator respectively,

$$\begin{aligned} E_1 &= E_1(m,k) = -\tfrac{1}{2}[cm+k+\tfrac{1}{2}(n-1)]^{-2} \\ E_2 &= E_2(m,k) = cm+2k+\tfrac{1}{2}n \end{aligned} \tag{4}$$

for all non-negative integers k.

Cisneros and McIntosh (1970) claim that their modifications give the same energy values but their equations lead to some unpleasant complications in the general case. However, most of their work is concerned with the two-dimensional case and then the two types of modification are identical.

3. Degeneracies of the excess systems

In the normal cases the degeneracy due to the angular momentum is increased by the 'extra' degeneracy between the m and k. The interest in excess angular momentum is to see how much of this 'accidental' degeneracy remains when c is not equal to 1. Clearly when c is irrational there is no extra degeneracy so that in this case, the excess angular momentum removes all the accidental degeneracy. Thus the important cases occur when $c = ab^{-1}$, where a and b are two mutually prime positive integers. In these cases the accidental degeneracy depends on the degeneracy of $(am+bk)$ and $(am+2bk)$.

As a and b are mutually prime every value of $am+bk$ can be uniquely written as $abE+aM+bK$, where E, M, K are positive integers and $M < b$, $K < a$. Hence the energy eigenvalues of the hydrogen atom with excess angular momentum can be uniquely written as:

$$E_1(m,k) = -\tfrac{1}{2}[(aE+ab^{-1}M+K)+\tfrac{1}{2}(n-1)]^{-2}, \tag{5}$$

and E_1 takes this value whenever $m = ib+M$ and $k = (E-i)a+K$ with $0 \leqslant i \leqslant E$.

If d is the highest common factor of a and $2b$, then $a' = ad^{-1}$ and $b' = 2bd^{-1}$ are mutually prime and $(cm+2k) = db^{-1}(a'm+b'k)$. Thus the energy values of the oscillator with excess angular momentum can be uniquely written as:

$$E_2(m,k) = 2(a'E+a'b'^{-1}M+K)+\tfrac{1}{2}n = 2(a'E+\tfrac{1}{2}ab^{-1}M+K)+\tfrac{1}{2}n \tag{6}$$

where E, M, K are positive integers $M < b'$, $K < a'$. E_2 takes this value whenever $m = ib'+M$, $k = (E-i)a'+K$ for $0 \leqslant i \leqslant E$.

The degeneracy with respect to each m is $D_n(m)$ and so for the hydrogen atom and oscillator respectively, the total degeneracies $D_1(E,M,K)$ and $D_2(E,M,K)$ of the energy levels determined by E, M, K are given by

$$\begin{aligned} D_1(E,M,K) &= \sum_{i=0}^{E} D_n(ib+M) \\ D_2(E,M,K) &= \sum_{i=0}^{E} D_n(i2bd^{-1}+M). \end{aligned} \tag{7}$$

4. The new degeneracies and O($n+1$), SU(n)

It is now possible to consider how the degeneracies of the excess systems depend on the values of a and b.

Firstly, when $a = b = 1$ both M and K are zero and $d = 1$. This is, of course, the standard case and using the properties of C_m^{m+n-1}, the degeneracies can be calculated from (2) and (7). It is not difficult to check that these expressions reduce to the usual formulae, which show that the degeneracies correspond to the dimensions of certain irreducible representations of O($n+1$) and SU(n), respectively. This well known property has been studied by many authors (eg Bander and Itzykson 1966).

Taking a to be greater than 1, while keeping $b = 1$, is equivalent to restricting the total orbital angular momentum, m, to being multiples of the integer a. As M is still zero the right-hand sides of (7) are exactly the same as when $a = 1$, except for d^{-1} in E_2. Hence for any integer a, the degeneracies of E_1 still correspond to O($n+1$). In fact for each $K < a$, the degeneracy $D_1(E, 0, K)$ equals the degeneracy of the normal hydrogen atom when the energy is $-\frac{1}{2}[E+\frac{1}{2}(n-1)]^{-2}$. Thus there are a series of degeneracy spaces, each series corresponding to the representations of O($n+1$) and so the degeneracy pattern resembles the direct sum of a copies of the normal hydrogen atom. This is very similar to the degeneracy patterns of the anisotropic harmonic oscillator with rationally related frequencies (King 1973).

When a is odd, $d = 1$ and so a similar result is true for the isotropic oscillator with excess angular momentum. Again there are a series of degeneracy spaces, each series corresponding to the totally symmetric irreducible representations of SU(n). But, if a is even, $d = 2$ and so $a' = \frac{1}{2}a$, which means that there are now $\frac{1}{2}a$ series of degeneracy spaces and within each series the dimensions of the degeneracy spaces correspond to the dimensions of representations of O($n+1$). So, in one sense, this system resembles the direct sum of $\frac{1}{2}a$ normal hydrogen atoms.

The right-hand sides of (7) are more complicated when $a = 1$ and b is greater than 1. However, if the degeneracies $D_1(E, M, 0)$ of all the energy levels with E fixed are summed as M varies from 0 to $b-1$ then the right-hand side includes all the $D_n(i)$, with $0 \leqslant i < (E+1)b$, exactly once. This is just the degeneracy of the energy value corresponding to $bE+b-1$ in the normal system. So in this case the excess angular momentum splits some of the normal degeneracy spaces of the hydrogen atom into b different subspaces.

A similar situation occurs in the oscillator case when the sum is taken over all even M varying from 0 to $2(b-1)$. But if the sum is taken over all M from 0 to b', when a is either 1 or 2, then the right-hand side of (7) includes all the $D_n(i)$ with $0 \leqslant i < (E+1)b'$ which gives the degeneracy corresponding to $Eb'+b'-1$ in the normal hydrogen atom.

Even when both a and b are greater than 1, the degeneracies are independent of K and a, except for the factor d. Thus, whenever a is greater than 1, there will be a copies of the degeneracies, in the sense that there will be a series of degeneracy spaces and each series contains exactly the same degeneracies, which can be found by holding b fixed and taking $a = 1, K = 0$. The only difference in the oscillator case, is that when a is even there are only $\frac{1}{2}a$ copies and these are found by taking $a = 2$.

Let $D_1^{a,b}(E, M, K)$ and $D_2^{a,b}(E, M, K)$ be the degeneracies of the energy level determined by E, M, K for the hydrogen atom and isotropic oscillator, where both systems have excess angular momentum determined by $c = ab^{-1}$. If $D_j(E) = D_j^{1,1}(E, 0, 0)$ then $D_j(E)$ is the degeneracy of the system without any excess. The comments of the previous

paragraphs can now be summarized in the formulae: for all $K < a$ and $j = 1, 2$,

$$D_j^{a,b}(E, M, K) = D_j^{a,b}(E, M, 0) = D_j^{1,b}(E, M, 0)$$

$$D_j^{a,1}(E, 0, K) = D_j(E)$$

$$\sum_{M=0}^{b-1} D_j^{1,b}(E, jM, 0) = D_j(bE + b - 1) \tag{8}$$

and

$$\sum_{M=0}^{b'-1} D_2^{1,b}(E, M, 0) = D_1(b'E + b' - 1)$$

except that when a is even and $K < \frac{1}{2}a$,

$$D_2^{a,b}(E, M, K) = D_1^{1,b}(E, M, 0).$$

These formulae clearly show how the introduction of a produces several copies of the same system, and when $b = 1$ this is just the normal system with the usual $0(n+1)$ and $SU(n)$ degeneracies. However, the effect of varying b is not clear because it is not obvious which energy values represented by the $E, M, 0$ are present on the left-hand sides of (8).

To establish a simple formula concerning the effect of b, let $D_j^{a,b}(E) = D_j^{a,b}(E', M, K)$ whenever $E = aE' + cM + K$, and then

$$\sum_{r=0}^{b-1} D_1^{a,b}(E - rb^{-1}) = D_1(\tilde{E}) \tag{10}$$

where $bE = a\tilde{E} + s$. It is easy to check that bE is always an integer and that as E', M, K vary bE takes all possible positive integral values. Hence $E - rb^{-1}$ is an energy value whenever E is, and so

$$bE - r = abE_r + aM_r + bK_r \tag{11}$$

where M_r, K_r are positive integers $M_r < b, K_r < a$.

Consider (11) for two values of r. The difference between r and r' is not a multiple of b and so $M_r = M_{r'}$ implies $r = r'$. Thus M_r takes all the values from 0 to $b-1$ as r ranges from 0 to $b-1$.

Let $bE = a\tilde{E} + s = a(be + G) + s$ where $0 \leqslant G < b, 0 \leqslant s < a$, and then substituting in (11) and rearranging,

$$ab(e - E_r) = a(M_r - G) + bK_r + r - s. \tag{12}$$

Now the right-hand side of (12) is less than $2ab$, but greater than $-ab$ so that E_r is either e or $e-1$. When $E_r = e$ the left-hand side is zero and so $M_r \leqslant G$, but when $E_r = e-1$ the left-hand side is ab, which is greater than $bK_r + r$ and so $M_r > G$.

Thus as r varies from 0 to $b-1$, M_r takes all the values from $G+1$ to $b-1$ when $E_r = e-1$, and all the values from 0 to G when $E_r = e$. Using the formulae (7), (8) and the definition of $D_1^{a,b}(E)$,

$$\sum_{r=0}^{b-1} D_1^{a,b}(E - rb^{-1}) = \sum_{r=0}^{b-1} D_1^{a,b}(E_r, M_r, 0) = \sum_{i=0}^{be+G} D_n(i) = D_1(be + G) = D_1(\tilde{E})$$

which proves the formula (10).

Thus when b is greater than 1, the introduction of excess angular momentum into the hydrogen atom has the interesting effect of splitting the degeneracy in such a way that

the sum of the degeneracies of any b consecutive levels gives the degeneracy of the normal system for a related level and so corresponds to a representation of $O(n+1)$.

The parallel results for the oscillator are given for all a by,

$$\sum_{r=0}^{b'-1} D_2^{a,b}(E-rb'^{-1}) = D_1(E^*)$$

where $b'E = aE^* + s$, and when a is odd by,

$$\sum_{r=0}^{b-1} D_2^{a,b}(E-rb^{-1}) = D_2(E^*)$$

where $2bE = a(2E^*+i)+s, i = 0$ or 1.

5. A decomposition of the hydrogen atom with excess angular momentum

It is possible (King 1973) to explain the degeneracies of the anisotropic harmonic oscillator, mentioned previously, by proving that the hamiltonian is equivalent to the direct sum of several isotropic hamiltonians. Unfortunately such a simple explanation does not seem possible for the systems with general excess angular momentum. One reason for this is that the factor c does not enter the problem linearly but has to be introduced for each value of m by the non-linear modification of $m(m+n-2)$ to $cm(cm+n-2)$. However, when $b=1$ the space on which the excess system acts can be decomposed, as will be shown for the hydrogen atom.

Each degeneracy space of the hydrogen atom with excess angular momentum can be decomposed into a set of subspaces, each of which is also a degeneracy space of the orbital angular momentum operator. Let $F_{m,k}$ be the subspace corresponding to the energy value $-\frac{1}{2}[am+k+\frac{1}{2}(n-1)]^{-2}$ on which the total orbital angular momentum takes the value m. For any fixed integer s, $0 \leqslant s < a$, define

$$V^s = \bigoplus_{m,k} F_{m,ak+s}$$

where the sum is over all possible m and k. There will be a distinct V^s formed as s varies and if V denotes the space on which the excess system acts, then

$$V = \bigoplus_s V^s,$$

where the direct sum contains each of the distinct V^s exactly once.

If $F'_{m,k}$ is the degeneracy subspace, with angular momentum m, of the ordinary hydrogen atom corresponding to the energy value $-\frac{1}{2}[m+k+\frac{1}{2}(n-1)]^{-2}$, then $F_{m,ak+s}$ and $F'_{m,k}$ have exactly the same dimension and so there is a unitary map $U^s_{m,k}$ transforming one into the other. This applies to each pair m, k, and so $U^s = \oplus_{m,k} U^s_{m,k}$ is a unitary map transforming V^s into V. Thus each V^s is isomorphic to V and if $U = \oplus_s U^s$, then $U: V \to \oplus_s V$.

Suppose $f_{m,k}$ belongs to $F'_{m,k}$, then $(U^s_{m,k})^{-1} f_{m,k}$ is an eigenvector of the excess system giving the value $-\frac{1}{2}[(am+ak+s)+\frac{1}{2}(n-1)]^{-2}$. Keeping s fixed this is clearly a function of the eigenvalue of $f_{m,k}$ in the normal system. Thus the action of the excess system on $(U^s)^{-1}V$ is equivalent to the action of a function of the ordinary hydrogen atom. Hence, under the unitary transformation U, the hydrogen atom with excess angular momentum

($b = 1$) is equivalent to the action of the direct sum of a simple functions of the ordinary hydrogen atom.

The degeneracies of these simple functions are clearly the same as those of the ordinary hydrogen atom and so this equivalence explains why the degeneracies of the excess system, with $b = 1$, must be equivalent to a copies of the degeneracies associated with the normal system. Further, this is similar to the decomposition of the anisotropic harmonic oscillator with rationally related frequencies, which was noted earlier.

The situation becomes much more complex when b is greater than 1. It is no longer possible to decompose the spaces simply because different values of m would be involved and the dimensions of the degeneracy spaces depend on m.

6. Conclusions

The degeneracies of the hydrogen atom and harmonic oscillator with excess angular momentum have been calculated and shown to be simply related to those occurring in the normal case. This relation has been explained for the hydrogen atom, when $b = 1$, by observing that the excess system can be decomposed into the direct sum of several normal systems. This decomposition is similar to that used with the anisotropic harmonic oscillator and is again defined in terms of the basis rather than being constructed elegantly. However, the results clearly demonstrate the important fact that even when some of the symmetry of the hydrogen atom and isotropic oscillator is destroyed by introducing unusual factors, the degeneracies can still be related to the representations of the groups $O(n+1)$ and $SU(n)$.

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